MATHEMATICS

Paper & Solution

Time: 3 Hrs. Max. Marks: 90

General Instructions:
(i) All questions are compulsory.
(ii) The question paper consists of 31 questions divided into four sections – A, B, C and D.
(iii) Section A contains 4 questions of 1 mark each. Section B contains 6 questions of 2 marks each, Section C contains 10 questions of 3 marks each and Section D contains 11 questions of 4 marks each.
(iv) Use of calculators is not permitted.

SECTION – A

Question numbers 1 to 4 carry 1 mark each.

1. If the quadratic equation $px^2 - 2\sqrt{5}px + 15 = 0$ has two equal roots then find the value of p.

Solution:

Given quadratic equation is,

$px^2 - 2\sqrt{5}px + 15 = 0$

Here, a = p, b = $2\sqrt{5}$ p, c= 15

For real equal roots, discriminant = 0

$\therefore b^2 - 4ac = 0$

$\therefore \left(2\sqrt{5}p\right)^2 - 4p(15) = 0$

$\therefore 20p^2 - 60p = 0$

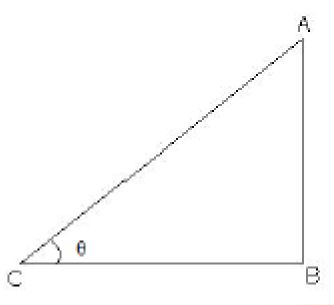
$\therefore 20p(p - 30) = 0$

$\therefore p = 30 \text{ or } p = 0$

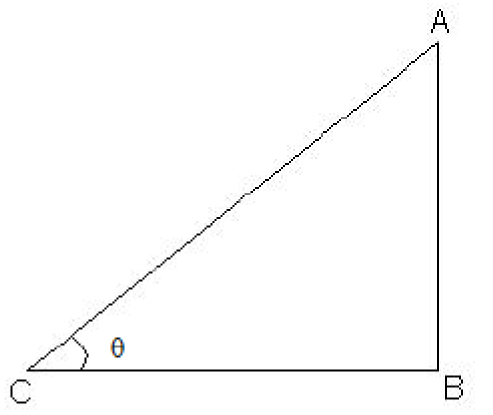
But, p = 0 is not possible.

$\therefore$ p = 30

2. In the following figure, a tower AB is 20 m high and BC, its shadow on the ground, is $20\sqrt{3}$ m long. Find the Sun's altitude.



Solution:
Let AB be the tower and BC be its shadow.



AB = 20, BC = 20$\sqrt{3}$

In ΔABC,

$\tan\theta = \frac{AB}{BC}$

$\tan\theta = \frac{20}{20\sqrt{3}}$

$\tan\theta = \frac{1}{\sqrt{3}}$

but, $\tan\theta = \frac{1}{\sqrt{3}}$

$\therefore \theta = 30$

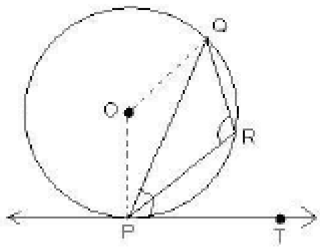
$\therefore$ The Sun is at an altitude of 30°.

3. Two different dice are tossed together. Find the probability that the product of the two numbers on the top of the dice is 6.

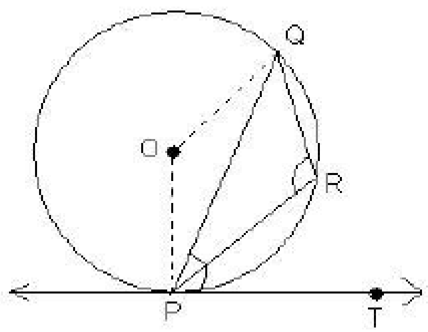
Solution:
Two dice are tossed
S = [(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),
(5,1),(5,2),(5,3),(5,4), (5,5),(5,6),
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)]
Total number of outcomes when two dice are tossed = 6 $\times$ 6= 36
Favourable events of getting product as 6 are:
(1$\times$ 6= 6), (6 $\times$1 =6),(2$\times$ 3= 6),(3$\times$ 2 =6)
i.e.(1,6), (6,1), (2,3), (3,2)
Favourable events of getting product as 6 = 4

$\therefore$ P(getting product as 6) $= \frac{4}{36} = \frac{1}{9}$

4. In the following figure, PQ is a chord of a circle with centre O and PT is a tangent. If ∠QPT = 60˚, find ∠PRQ



Solution:



m∠OPT 90° ($\therefore$ radius is perpendicular to the tangent)
So, ∠OPQ = ∠OPT - ∠QPT
= 90° - 60°
= 30°
m∠POQ = 2∠QPT 2 × 60° =120°
reflex m∠POQ = 360° - 120° =240°

$m\angle PRQ = \frac{1}{2}\text{reflex}\angle POQ$

$= \frac{1}{2} \times 240°$

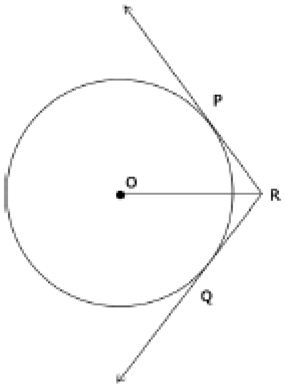
$= 120°$

$\therefore \angle PRQ = 120°$

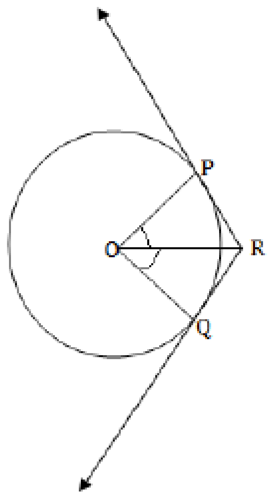
SECTION B

Question numbers 5 to 10 carry 2 marks each.

5. In the following figure, two tangents RQ and RP are drawn from an external point R to the circle with centre O, If ∠PRQ = 120˚, then prove that OR = PR + RQ.



Solution:



Given that m$\angle$PRQ = 120°

We know that the line joining the centre and the external point is the angle bisector between the tangents.

Thus, $m\angle PRO = m\angle QRO = \frac{120^\circ}{2} = 60^\circ$

Also we know that lengths of tangents from an external point are equal.

Thus, PR = RQ.

Join OP and OQ.

Since OP and OQ are the radii from the centre O,

OP $\perp$ PR and OQ $\perp$ RQ.

Thus, ΔOPR and ΔOQR are right angled congruent triangles.

Hence, $\angle POR = 90^\circ - \angle PRO = 90^\circ - 60^\circ = 30^\circ$

$\angle QOR = 90^\circ - \angle QRO = 90^\circ - 60^\circ = 30^\circ$

$\sin\angle QRO = \sin 30^\circ = \frac{1}{2}$

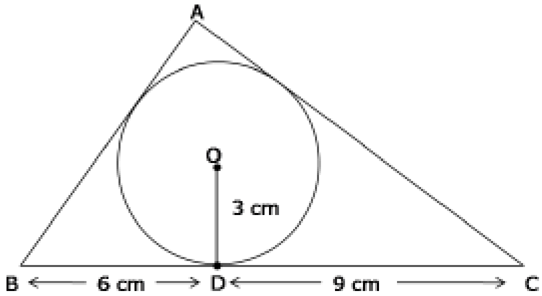
$\frac{PR}{OR} = \frac{1}{2}$

Thus, $\Rightarrow OR = 2PR$

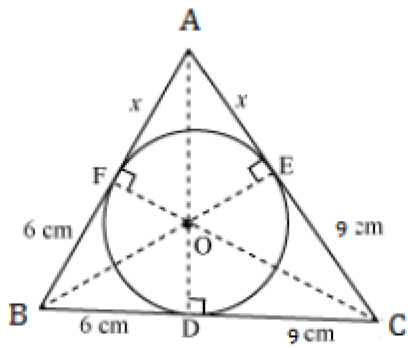
$\Rightarrow OR = PR + PR$

$\Rightarrow OR = PR + QR$

6. In Figure 4, a ΔABC is drawn to circumscribe a circle of radius 3 cm, such that the segments BD and DC are respectively of lengths 6 cm and 9 cm. If the area of Δ ABC is 54 cm^2, then find the lengths of sides AB and AC.



Solution:



Let the given circle touch the sides AB and AC of the triangle at points F and E respectively and let the length of line segment AF be *x*.

Now, it can be observed that:

BF = BD = 6 cm (tangents from point B)

CE = CD = 9 cm (tangents from point C)

AE = AF = x (tangents from point A)

AB = AF + FB = x + 6

BC = BD + DC = 6 + 9 = 15

CA = CE + EA = 9 + x

2s = AB + BC + CA = x + 6 + 15 + 9 + x = 30 + 2x

s = 15 + x

s – a = 15 + x – 15 = x

s – b = 15 + x – (x + 9) = 6

s – c = 15 + x – (6 + x) = 9

Area of Δ ABC $\sqrt{s(s-a)(s-b)(s-c)}$

$$54 = \sqrt{(15+x)(x)(6)(9)}$$

$$54 = 3\sqrt{6(15x+x^2)}$$

$$18 = \sqrt{6(15x+x^2)}$$

$$324 = 6(15x+x^2)$$

$$54 = 15x+x^2$$

$x^2 + 15x - 54 = 0$

$x^2 + 18x - 3x - 54 = 0$

$x(x+18) - 3(x+18)$

$(x+18)(x-3) = 0$

$x = -18$ and $x = 3$

As distance cannot be negative, x = 3

AC = 3 + 9 = 12

AB = AF + FB = 6 + x = 6 + 3 = 9

7. Solve the following quadratic equation for x:

$4x^2 + 4bx - (a^2 - b^2) = 0$

Solution:

$$4x^2 + 4bx - (a^2 - b^2) = 0$$

$$\Rightarrow x^2 + bx - \left(\frac{a^2 - b^2}{4}\right) = 0$$

$$\Rightarrow x^2 + 2\left(\frac{b}{2}\right)x = \frac{a^2 - b^2}{4}$$

$$\Rightarrow x^2 + 2\left(\frac{b}{2}\right)x + \left(\frac{b}{2}\right)^2 = \frac{a^2 - b^2}{4} + \left(\frac{b}{2}\right)^2$$

$$\Rightarrow \left(x + \frac{b}{2}\right)^2 = \frac{a^2}{4}$$

$$\Rightarrow x + \frac{b}{2} = \pm\frac{a}{2}$$

$$\Rightarrow x = \frac{-b}{2} \pm \frac{a}{2}$$

$$\Rightarrow x = \frac{-b-a}{2}, \frac{-b+a}{2}$$

Hence, the roots are $-\left(\frac{a+b}{2}\right)$ and $\left(\frac{a-b}{2}\right)$.

8. In an AP, if $S_5 + S_7 = 167$ and $S_{10} = 235$, then find the A.P., where S_n denotes the sum of its first n terms

Solution:

$S_5 + S_7 = 167$ and $S_{10} = 235$

Now, $S_n = \frac{n}{2}\{2a + (n-1)d\}$

$\therefore S_5 + S_7 = 167$

$$\Rightarrow \frac{5}{2}\{2a + 4d\} + \frac{7}{2}\{2a + 6d\} = 167$$

$\Rightarrow 5a + 10d + 7a + 21d = 167$

$\Rightarrow 12a + 31d = 167$ ....(1)

Also, $S_{10} = 235$

$\therefore \frac{10}{2}\{2a+9d\}=235$

$\Rightarrow 10a+45d=235$

$\Rightarrow 2a+9d=47$ ....(2)

Multiplying equation (2) by 6, we get

12a + 54d = 282 ....(3)

Subtracting (1) from (3), we get

$$\begin{array}{r} 12a+54d=282 \\ (-)12a+31d=167 \\ -\quad\ -\quad\ - \\ \hline 23d=115 \end{array}$$

$\therefore d=5$

Substituting value of d in (2), we have

2a + 9(5)= 47

$\Rightarrow$ 2a + 45= 47

$\Rightarrow$ 2a = 2

$\Rightarrow$ a = 1

Thus, the given A.P. is 1, 6, 11, 16,..........

9. The points A(4, 7), B(p, 3) and C(7, 3) are the vertices of a right triangle, right-angled at B, Find the values of P.

Solution:

Δ ABC is right angled at B.

$\therefore AC^2=AB^2+BC^2$ ....(1)

Also, $A\equiv(4,7), B\equiv(p,3)$ and $C\equiv(7,3)$

Now, $AC^2=(7-4)^2+(3-7)^2=(3)^2+(-4)^2=9+16=25$

$AB^2=(p-4)^2+(3-7)^2=p^2-8p+16+(-4)^2$

$=p^2-8p+16+16$

$=p^2-8p+32$

$BC^2=(7-p)^2+(3-3)^2=49-14p+p^2+0$

$=p^2-14p+49$

From (1), we have

25= (p^2 - 8p + 32)+ (p^2 -14p +49)

$\Rightarrow$ 25 = $2p^2$- 22p +81

$\Rightarrow 2p^2$ - 22p +56= 0

$\Rightarrow p^2$ - 11p + 28 = 0

$\Rightarrow p^2$ - 7p - 4p+ 28 = 0

$\Rightarrow$ p(p -7) – 4(p – 7) = 0

$\Rightarrow$ (p – 7) (p – 4) = 0

$\Rightarrow$ p = 7 and p = 4

10. Find the relation between x and y if the points A(x, y), B(-5, 7) and C(-4, 5) are collinear.

Solution:

Given, the points A(x,y), B(-5,7) and C(-4,5) are collinear.
So, the area formed by these vertices is 0.

$$\therefore \frac{1}{2}\left[x(7-5)+(-5)(5-y)+(-4)(y-7)\right]=0$$

$$\Rightarrow \frac{1}{2}\left[2x-25+5y-4y+28\right]=0$$

$$\Rightarrow \frac{1}{2}\left[2x+y+3\right]=0$$

$$\Rightarrow 2x+y+3=0$$

$$\Rightarrow y=-2x-3$$

SECTION C

Question numbers 11 to 20 carry 3 marks each.

11. The 14th term of an A.P. is twice its 8th term. If its 6th term is -8, then find the sum of its first 20 terms.

Solution:

Here it is given that,
$T_{14} = 2(T_8)$
$\Rightarrow$ a + (14 – 1)d = 2[a + (8 – 1)d]
$\Rightarrow$ a + 13d = 2[a + 7d]
$\Rightarrow$ a + 13d = 2a + 14d
$\Rightarrow$ 13d – 14d = 2a – a
$\Rightarrow$ – d = a .... (1)
Now, it is given that its 6^{th} term is –8.
$T_6 = -8$
$\Rightarrow$ a + (6 – 1)d = – 8
$\Rightarrow$ a + 5d = –8
$\Rightarrow$ –d + 5d = –8 [$\because$ Using (1)]
$\Rightarrow$ 4d = –8
$\Rightarrow$ d = –2
Subs. this in eq. (1), we get a = 2
Now, the sum of 20 terms,

$$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$$

$$S_{20}=\frac{20}{2}\left[2a+(20-1)d\right]$$

= 10 [2(2) + 19(-2)]
= 10[4 - 38]
= -340

12. Solve for x:

$$\sqrt{3}x^2-2\sqrt{2}x-2\sqrt{3}=0$$

Solution:

For the given equation, $\sqrt{3}x^2-2\sqrt{2}x-2\sqrt{3}=0$
Comparing this equation with $ax^2 + bx + c = 0$, we obtain

$a=\sqrt{3}, b=-2\sqrt{2}, c=-2\sqrt{3}$

Now, $D=\sqrt{b^2-4ac}$

$=\sqrt{(-2\sqrt{2})^2-4\left(4\sqrt{3}\right)\left(-2\sqrt{3}\right)}$

$=\sqrt{8+24}=\sqrt{32}=4\sqrt{2}$

Using quadratic formula, we obtain

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-(-2\sqrt{2})\pm 4\sqrt{2}}{2\sqrt{3}}$$

$$\Rightarrow x=\frac{2\sqrt{2}+4\sqrt{2}}{2\sqrt{3}} \text{ or } x=\frac{2\sqrt{2}-4\sqrt{2}}{2\sqrt{3}}$$

$$\Rightarrow x=\frac{\sqrt{2}+2\sqrt{2}}{\sqrt{3}} \text{ or } x=\frac{\sqrt{2}-2\sqrt{2}}{\sqrt{3}}$$

$$\Rightarrow x=\frac{3\sqrt{2}}{\sqrt{3}} \text{ or } x=\frac{-\sqrt{2}}{\sqrt{3}}$$

$$\Rightarrow x=\sqrt{3}\sqrt{2} \text{ or } x=\frac{-\sqrt{2}}{\sqrt{3}}$$

$$\therefore x=\sqrt{6} \text{ or } x=\frac{-\sqrt{2}}{\sqrt{3}}$$

13. The angle of elevation of an aeroplane from point A on the ground is 60°. After flight of 15 seconds, the angle of elevation changes to 30°. If the aeroplane is flying at a constant height of $1500\sqrt{3}$ m, find the speed of the plane in km/hr.

Solution:

Let BC be the height at watch the aeroplane is observed from point A.

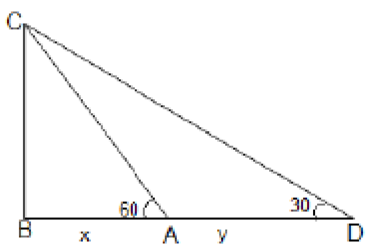
Then, BC = 1500 $\sqrt{3}$

In 15 seconds, the aeroplane moves from point A to D.

A and D are the points where the angles of elevation 60° and 30° are formed respectively.

Let BA = x metres and AD = y metres

BC = x + y



In ΔCBA,

$$\tan 60^\circ = \frac{BC}{BA}$$

$$\sqrt{3} = \frac{1500\sqrt{3}}{x}$$

$\therefore x = 1500\ m$ ....(1)

In ΔCBD,

$$\tan 30^\circ = \frac{BC}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{1500\sqrt{3}}{x+y}$$

$\therefore x + y = 1500(3) = 4500$

$\therefore 1500 + y = 4500$

$\therefore y = 3000\ m$ ....(2)

We know that, the aeroplane moves from point A to D in 15 seconds and the distance covered is 3000 metres. (by 2)

$$\text{Speed } = \frac{\text{distance}}{\text{time}}$$

$$\text{Speed } = \frac{3000}{15}$$

Speed 200m/s

Converting it to km/hr $= 200 \times \frac{18}{5} = 720 km / hr$

14. If the coordinates of points A and B are (-2, -2) and (2, -4) respectively, find the coordinates of P such that AP = $\frac{3}{7}$ AB, where P lies on the line segment AB.

Solution:

Here, P(x,y) divides line segment AB, such that

$$AP = \frac{3}{7}AB$$

$$\frac{AP}{AB} = \frac{3}{7}$$

$$\frac{AB}{AP} = \frac{7}{3}$$

$$\frac{AB}{AP} - 1 = \frac{7}{3} - 1$$

$$\frac{AB - AP}{AP} = \frac{7-3}{3}$$

$$\frac{BP}{AP} = \frac{4}{3}$$

$$\frac{AP}{BP} = \frac{3}{4}$$

$\therefore$ P divides AB in the ratio 3: 4

$$x = \frac{3\times2+4(-2)}{3+4}; y = \frac{3\times(-4)+4(-2)}{3+4}$$

$$x = \frac{6-8}{7}; y = \frac{-12-8}{7}$$

$$x = \frac{-2}{7}; y = \frac{-20}{7}$$

$\therefore$ The coordinates of P are $\left(\frac{-2}{7}, \frac{-20}{7}\right)$

15. The probability of selecting a red ball at random from a jar that contains only red, blue and orange balls is $\frac{1}{4}$. The probability of selecting a blue ball at random from the same jar $\frac{1}{3}$. If the jar contains 10 orange balls, find the total number of balls in the jar.

Solution:

Here the jar contains red, blue and orange balls.

Let the number of red balls be x.

Let the number of blue balls be y.

Number of orange balls = 10

Total number of balls = x + y + 10

Now, let P be the probability of drawing a ball from the jar

P(a red ball) $= \frac{x}{x+y+10}$

$$\Rightarrow \frac{1}{4} = \frac{x}{x+y+10}$$

$$\Rightarrow 4x = x+y+10$$

$$\Rightarrow 3x - y = 10 \quad ....(i)$$

Next,

P (a blue ball) $= \frac{y}{x+y+10}$

$$\Rightarrow \frac{1}{3} = \frac{y}{x+y+10}$$

$$\Rightarrow 3y = x+y+10$$

$$\Rightarrow 2y - x = 10 \quad ....(ii)$$

Multiplying eq. (i) by 2 and adding to eq. (ii), we get

$$6x - 2y = 20$$

$$-x + 2y = 10$$

$$5x = 30$$

$$\Rightarrow x = 6$$

Subs. x = 6 in eq. (i), we get y = 8

$\therefore$ Total number of balls = x + y + 10 = 6 + 8 + 10 = 24

Hence, total number of balls in the jar is 24.

16. Find the area of the minor segment of a circle of radius 14 cm, when its central angle is 60˚. Also find the area of the corresponding major segment. [Use $\pi = \frac{22}{7}$]

Solution:
Radius of the circle =14 cm
Central Angle, θ = 60 ,
Area of the minor segment

$$= \frac{\theta}{360^\circ} \times \pi r^2 - \frac{1}{2} r^2 \sin\theta$$

$$= \frac{60^\circ}{360^\circ} \times \pi \times 14^2 - \frac{1}{2} \times 14^2 \times \sin 60^\circ$$

$$= \frac{1}{6} \times \frac{22}{7} \times 14 \times 14 - \frac{1}{2} \times 14 \times 14 \times \frac{\sqrt{3}}{2}$$

$$= \frac{22 \times 14}{3} - 49\sqrt{3}$$

$$= \frac{22 \times 14}{3} - \frac{147\sqrt{3}}{3}$$

$$= \frac{308 - 147\sqrt{3}}{3} \text{cm}^2$$

Area of the minor segment $= \frac{308 - 147\sqrt{3}}{3} cm^2$

17. Due to sudden floods, some welfare associations jointly requested the government to get 100 tents fixed immediately and offered to contribute 50% of the cost. If the lower part of each tent is of the form of a cylinder of diameter 4.2 m and height 4 m with the conical upper part of same diameter but height 2.8 m, and the canvas to be used costs Rs. 100 per sq. m, find the amount, the associations will have to pay. What values are shown by these associations? [Use $\pi = \frac{22}{7}$]

Solution:
Diameter of the tent = 4.2 m
Radius of the tent, r = 2.1 m
Height of the cylindrical part of tent, $h_{cylinder}$ = 4 m
Height of the conical part, h_{cone} = 2.8 m
Slant height of the conical part, l

$$= \sqrt{h_{cone}{}^2 + r^2}$$

$$= \sqrt{2.8^2 + 2.1^2}$$

$$= \sqrt{2.8^2 + 2.1^2}$$

$$= \sqrt{12.25} = 3.5\,m$$

Curved surface area of the cylinder = 2 π r $h_{cylinder}$

$$= 2 \times \frac{22}{7} \times 2.1 \times 4$$

$$= 22 \times 0.3 \times 8 = 52.8\,m^2$$

Curved surface area of the conical tent $=\pi rl=\frac{22}{7}\times 2.1\times 3.5=23.1\,m^2$

Total area of cloth required for building one tent

= Curved surface area of the cylinder + Curved surface area of the conical tent

= 52.8 + 23.1

= 75.9 m^2

Cost of building one tent = 75.9 × 100 = Rs 7590

Total cost of 100 tents = 7590 × 100 = Rs 7,59,000

Cost to be borne by the associations $=\frac{759000}{2}$ = Rs. 3,79,500

It shows the helping nature, unity and cooperativeness of the associations.

18. A hemispherical bowl of internal diameter 36 cm contains liquid. This liquid is filled into 72 cylindrical bottles of diameter 6 cm. Find the height of each bottle, if 10% liquid is wasted in this transfer.

Solution:

Internal diameter of the bowl = 36 cm

Internal radius of the bowl, r = 18 cm

Volume of the liquid, $V=\frac{2}{3}\pi r^3=\frac{2}{3}\times\pi\times 18^3$

Let the height of the small bottle be 'h'.

Diameter of a small cylindrical bottle = 6 cm

Radius of a small bottle, R = 3 cm

Volume of a single bottle = $\pi R^2h = \pi\times 3^2\times h$

No. of small bottles, n = 72

Volume wasted in the transfer $=\frac{10}{100}\times\frac{2}{3}\times\pi\times 18^3$

Volume of liquid to be transferred in the bottles

$=\frac{2}{3}\times\pi\times 18^3-\frac{10}{100}\times\frac{2}{3}\times\pi\times 18^3$

$=\frac{2}{3}\times\pi\times 18^3\left(1-\frac{10}{100}\right)$

$=\frac{2}{3}\times\pi\times 18^3\times\frac{90}{100}$

Number of small cylindrical bottles $=\frac{\text{Volume of the liquid to be transferred}}{\text{Volume of a single bottle}}$

$\Rightarrow 72=\frac{\frac{2}{3}\times\pi\times 18^3\times\frac{90}{100}}{\pi\times 3^2\times h}$

$\Rightarrow 72=\frac{\frac{2}{3}\times 18^3\times\frac{9}{10}}{3^2\times h}$

$\Rightarrow h=\frac{\frac{2}{3}\times 18\times 18\times 18\times\frac{9}{10}}{3^2\times 72}$

$\therefore h=5.4\,cm$

Height of the small cylindrical bottle = 10.8 cm

19. A cubical block of side 10 cm is surmounted by a hemisphere. What is the largest diameter that the hemisphere can have? Find the cost of painting the total surface area of the solid so formed, at the rate of Rs. 5 per sq. cm. [Use π = 3.14]

Solution:

Side of the cubical block, a = 10 cm

Longest diagonal of the cubical block = a $\sqrt{3}$ = 10 $\sqrt{3}$ cm

Since the cube is surmounted by a hemisphere, therefore the side of the cube should be equal to the diameter of the hemisphere.

Diameter of the sphere = 10 cm

Radius of the sphere, r = 5 cm

Total surface area of the solid = Total surface area of the cube – Inner cross-section area of the hemisphere + Curved surface area of the hemisphere

$= 6a^2 - \pi r^2 + 2 + \pi r^2$

$= 6a^2 + \pi r^2$

$= 6 \times (10)^2 + 3.14 \times 5^2$

$= 600 + 78.5 = 678.5\ cm^2$

Total surface area of the solid = 678.5 cm^2

20. 504 cones, each of diameter 3.5 cm and height 3 cm, are melted and recast into a metallic sphere, Find the diameter of the sphere and hence find its surface area. [Use $\pi = \frac{22}{7}$]

Solution:

No. of cones = 504

Diameter of a cone = 3.5 cm

Radius of the cone, r = 1.75 cm

Height of the cone, h = 3 cm

Volume of a cone

$= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times \left(\frac{3.5}{2}\right)^2 \times 3$

$= \frac{1}{3} \times \pi \times \frac{3.5}{2} \times \frac{3.5}{2} \times 3\ cm^3$

Volume of 504 cones

$= 504 \times \frac{1}{4} \times \pi \times \frac{3.5}{2} \times \frac{3.5}{2} \times 3\ cm^3$

Let the radius of the new sphere be 'R'.

Volume of the sphere $= \frac{4}{3}\pi R^3$

Volume of 504 cones = Volume of the sphere

$$504\times\frac{1}{3}\times\pi\times\frac{3.5}{2}\times\frac{3.5}{2}\times3=\frac{4}{3}\pi R^3$$
$$\Rightarrow\frac{504\times1\times\pi\times3.5\times3.5\times3\times3}{3\times2\times2\times4\times\pi}=R^3$$
$$\Rightarrow R^3=\frac{504\times3\times49}{64}$$
$$\Rightarrow R^3=\frac{7\times8\times9\times3\times7^2}{64}$$
$$\Rightarrow R^3=\frac{8\times27\times7^3}{64}$$
$$\Rightarrow R=\frac{2\times3\times7}{4}$$
$$\therefore R=\frac{21}{2}=10.5\,cm$$
$\therefore$ Radius of the new sphere = 10.5 cm

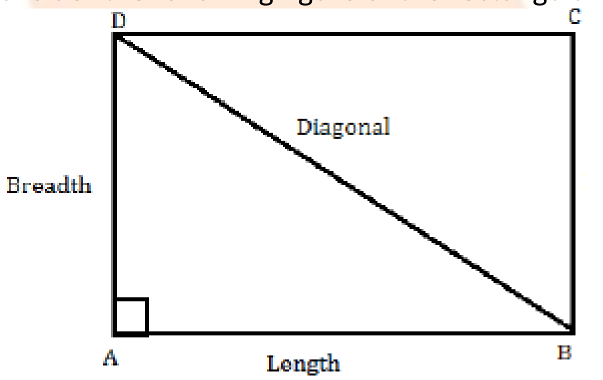
SECTION D

Question numbers 21 to 31 carry 4 marks each.

21. The diagonal of a rectangular field is 16 metres more than the shorter side. If the longer side is 14 metres more than the shorter side, then find the lengths of the sides of the field.

Solution:

Let l be the length of the longer side and b be the length of the shorter side.
Given that the length of the diagonal of the rectangular field is 16 metres more than the shorter side.
Thus, diagonal = 16 + b
Since longer side is 14 metres more than shorter side, we have,
l = 14 + b
Diagonal is the hypotenuse of the triangle.
Consider the following figure of the rectangular field.



By applying Pythagoras Theorem in Δ ABD, we have,
Diagonal^2 = Length^2 = Breadth^2
$$\Rightarrow(16+b)^2=(14+b)^2+b^2$$
$$\Rightarrow256+b^2+32b=196+b^2+28b+b^2$$
$$\Rightarrow256+32b=196+28b+b^2$$
$$\Rightarrow60+32b=28b+b^2$$

$\Rightarrow b^2 - 4b - 60 = 0$

$\Rightarrow b^2 - 10b + 6b - 60 = 0$

$\Rightarrow b(b-10) + 6(b-10) = 0$

$\Rightarrow (b+6)(b-10) = 0$

$\Rightarrow (b+6) = 0 \text{ or } (b-10) = 0$

$\Rightarrow b = -6 \text{ or } b = 10$

As breadth cannot be negative, breadth = 10 m
Thus, length of the rectangular field = 14 + 10 = 24 m

22. Find the 60th term of the AP 8, 10, 12, ..., if it has a total of 60 terms and hence find the sum of its last 10 terms.
Solution:
Consider the given A.P. 8, 10, 12, ...
Here the initial term is 8 and the common difference is 10 - 8 = 2 and 12 - 10 = 2
General term of an A.P. is t_n and formula to find out t_n is

$t_n = a + (n-1)d$

$\Rightarrow t_{60} = 8 + (60-1) \times 2$

$\Rightarrow t_{60} = 8 + 59 \times 2$

$\Rightarrow t_{60} = 8 + 118$

$\Rightarrow t_{60} = 126$

We need to find the sum of the last 10 terms.
Thus,
Sum of last 10 terms = Sum of first 60 terms - Sum of first 50 terms

$S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow S_{60} = \frac{60}{2}[2 \times 8 + (60-1) \times 2]$

$\Rightarrow S_{60} = 30[16 + 59 \times 2]$

$\Rightarrow S_{60} = 30[134]$

$\Rightarrow S_{60} = 4020$

Similarly,

$\Rightarrow S_{50} = \frac{50}{2}[2 \times 8 + (50-1) \times 2]$

$\Rightarrow S_{50} = 25[16 + 49 \times 2]$

$\Rightarrow S_{50} = 25[114]$

$\Rightarrow S_{50} = 2850$

Thus the sum of last 10 terms $= S_{60} - S_{50} = 4020 - 2850 = 1170$
Therefore,
Sum of last 10 terms = Sum of first 60 terms - Sum of first 50 terms

23. A train travels at a certain average speed for a distance of 54 km and then travels a distance of 63 km at an average speed of 6 km/h more than the first speed. If it takes 3 hours to complete the total journey, what is its first speed?

Solution:

Let x be the first speed of the train.

We know that $\frac{\text{Distance}}{\text{Speed}} = \text{time}$

Thus, we have,

$$\frac{54}{x}+\frac{63}{x+6}=3\text{ hours}$$

$$\Rightarrow \frac{54(x+6)+63x}{x(x+6)}=3$$

$$\Rightarrow 54(x+6)+63x=3x(x+6)$$

$$\Rightarrow 54x+324+63x=3x^2+18x$$

$$\Rightarrow 117x+324=3x^2+18x$$

$$\Rightarrow 3x^2-117x-324+18x=0$$

$$\Rightarrow 3x^2-99x-324=0$$

$$\Rightarrow x^2-33x-108=0$$

$$\Rightarrow x^2-36x+3x-108=0$$

$$\Rightarrow x(x-36)+3(x-36)=0$$

$$\Rightarrow (x+3)(x-36)=0$$

$$\Rightarrow (x+3)=0 \text{ or } (x-36)=0$$

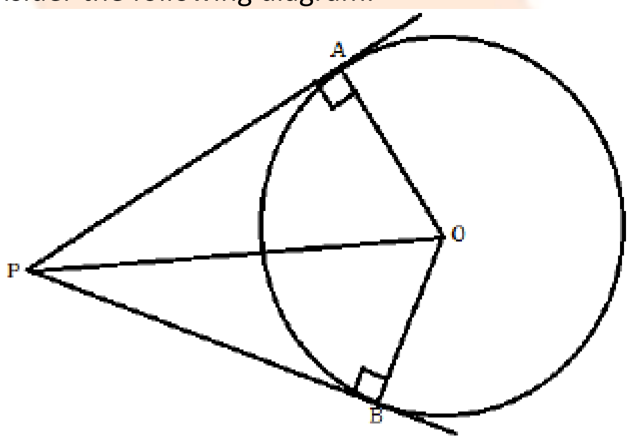
$$\Rightarrow x=-3 \text{ or } x=36$$

Speed cannot be negative and hence initial speed of the train is 36 km/hour.

24. Prove that the lengths of the tangents drawn from an external point to a circle are equal.

Solution:

Consider the following diagram.



Let P be an external point and PA and PB be tangents to the circle.

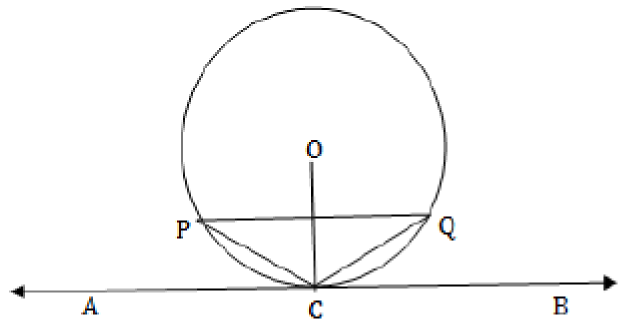
We need to prove that PA = PB

Now consider the triangles Δ OAP and Δ OBP

m$\angle$A = m$\angle$B = 90

OP = OP [common]
OA = OB = radii of the circle
Thus, by Right Angle-Hypotenuse-Side criterion of congruence we have,
$\Delta OAP \cong \Delta OBP$
The corresponding parts of the congruent triangles are congruent.
Thus,
PA = PB

25. Prove that the tangent drawn at the mid-point of an arc of a circle is parallel to the chord joining the end points of the arc.
Solution:
In the figure, C is the midpoint of the minor arc PQ, O is the centre of the circle and
AB is tangent to the circle through point C.
We have to show the tangent drawn at the midpoint of the arc PQ of a circle is parallel to the chord joining the end points of the arc PQ.
We will show PQ || AB.
It is given that C is the midpoint point of the arc PQ.
So, arc PC = arc CQ.
$\Rightarrow$ PC = CQ



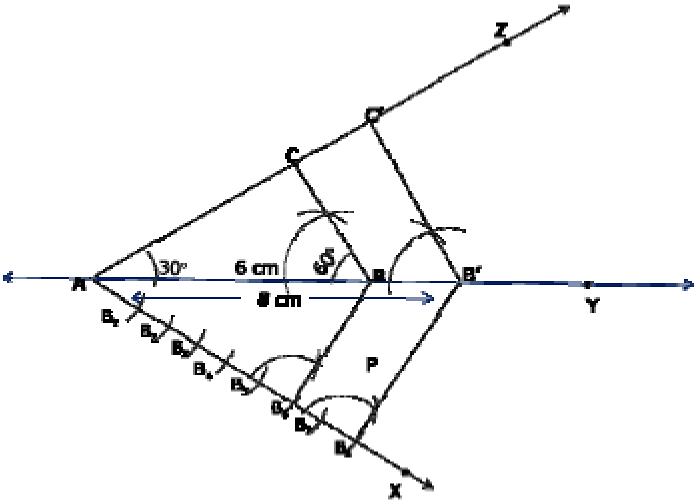
This shows that ΔPQC is an isosceles triangle.
Thus, the perpendicular bisector of the side PQ of ΔPQC passes through vertex C.
The perpendicular bisector of a chord passes through the centre of the circle.
So the perpendicular bisector of PQ passes through the centre O of the circle.
Thus perpendicular bisector of PQ passes through the points O and C.
$\Rightarrow PQ \perp OC$
AB is the tangent to the circle through the point C on the circle.
$\Rightarrow AB \perp OC$
The chord PQ and the tangent PQ of the circle are perpendicular to the same line OC.
$\therefore$ PQ || AB.

26. Construct a ΔABC in which AB = 6 cm, $\angle A = 30°$ and $\angle B = 60°$, Construct another $\Delta AB'C'$ similar to ΔABC with base AB' = 8 cm.
Solution:
Construct the ΔABC as per given measurements.
In the half plane of $\overline{AB}$ which does not contain C, draw $\overrightarrow{AX}$ such that $\angle BAX$ is an acute angle.
3) With some appropriate radius and centre A, Draw an arc to intersect $\overrightarrow{AX}$ at B_1. Similarly, with centre B_1 and the same radius, draw an arc to intersect $\overrightarrow{BX}$ at B_2 such that $B_1B_2 = B_3B_4 = B_4B_5 = B_5B_6 = B_6B_7 = B_7B_8$

4) Draw $\overline{B_6B}$.

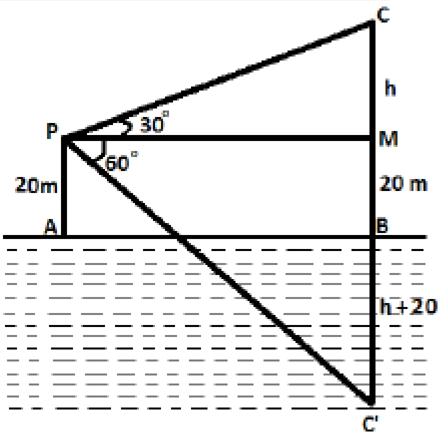
5) Through B_8 draw a ray parallel to $\overline{B_6B}$. to intersect $\overline{AY}$ at B'.

6) Through B' draw a ray parallel to $\overline{BC}$ to intersect $\overline{AZ}$ at C'.
Thus, ΔAB'C' is the required triangle.



27. At a point A, 20 metres above the level of water in a lake, the angle of elevation of a cloud is 30°. The angle of depression of the reflection of the cloud in the lake, at a A is 60°. Find the distance of the cloud from A.

Solution:



Let AB be the surface of the lake and P be the point of observation such that
AP = 20 metres. Let C be the position of the cloud and C' be its reflection in the lake.
Then CB = C'B. Let PM be perpendicular from P on CB.
Then m$\angle$CPM=30 and m$\angle$C'PM = 60
Let CM = h. Then CB = h + 20 and C'B = h + 20.
In CMP we have,

$\tan 30^\circ = \frac{CM}{PM}$

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{PM}$

$\Rightarrow PM = \sqrt{3}h$..................(i)

In Δ PMC' we have,

$\tan 60^\circ = \frac{C'M}{PM}$

$\Rightarrow \sqrt{3} = \frac{C'B + BM}{PM}$

$\Rightarrow \sqrt{3} = \frac{h + 20 + 20}{PM}$

$\Rightarrow PM = \frac{h + 20 + 20}{\sqrt{3}}$..................(ii)

From equation (i) and (ii), we get

$\sqrt{3}h = \frac{h + 20 + 20}{\sqrt{3}}$

$\Rightarrow 3h = h + 40$

$\Rightarrow 2h = 40$

$\Rightarrow h = 20\,\text{m}$

Now, CB = CM + MB = h + 20 + 20 + 20 = 40.

Hence, the height of the cloud from the surface of the lake is 40 metres.

28. A card is drawn at random from a well-shuffled deck of playing cards. Find the probability that the card drawn is

i. a card of spade or an ace.

ii. a black king.

iii. neither a jack nor a king

iv. either a king or a queen.

Solution:

Let S be the sample space of drawing a card from a well-shuffled deck.

$n(S) = {}^{52}C_1 = 52$

(i)There are 13 spade cards and 4 ace's in a deck As ace of spade is included in 13 spade cards, so there are 13 spade cards and 3 ace's

a card of spade or an ace can be drawn in ${}^{13}C_1 + {}^{3}C_1 = 13 + 3 = 16$

Probability of drawing a card of spade or an ace $= \frac{16}{52} = \frac{4}{13}$

(ii)There are 2 black King cards in a deck a card of black King can be drawn in ${}^{2}C_1 = 2$

Probability of drawing a black king $= \frac{2}{52} = \frac{1}{26}$

(iii) There are 4 Jack and 4 King cards in a deck.

So there are 52 - 8 = 44 cards which are neither Jacks nor Kings. a card which is neither a Jack nor a King can be drawn in $^{44}C_1 = 44$

Probability of drawing a card which is neither a Jack nor a King = $\frac{44}{52} = \frac{11}{13}$

(iv)There are 4 King and 4 Queen cards in a deck.
So there are 4 - 4 = 8 cards which are either King or Queen.
a card which is either a King or a Queen can be drawn in $^{8}C_1 = 8$

Probability of drawing a card which is either a King or a Queen = $\frac{8}{52} = \frac{2}{13}$

29. Find the values of k so that the area of the triangle with vertices (1, -1), (-4, 2k) and (-k, -5) is 24 sq. units.

Solution:

Take $(x_1, y_1) = (1, -1), (-4, 2k)$ and $(-k, -5)$

It is given that the area of the triangle is 24 sq. unit

Area of the triangle having vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$\therefore 24 = \frac{1}{2}[1(2k - (-5)) + (-4)((-5) - (-1)) + (-k)((-1) - 2k)]$$

$$48 = \left[(2k + 5) + 16 + (k + 2k^2)\right]$$

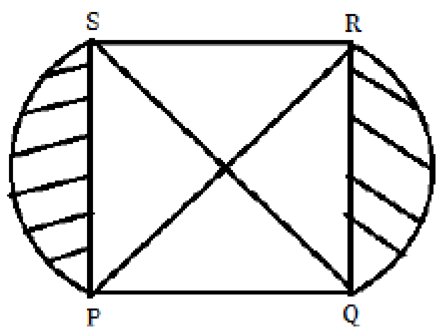
$$\therefore 2k^2 + 3k - 27 = 0$$

$$\therefore (2k + 9)(k - 3) = 0$$

$$\therefore k = -\frac{9}{2} \text{ or } k = 3$$

$\therefore$ The values of k are $-\frac{9}{2}$ and 3.

30. In the following figure, PQRS is square lawn with side PQ = 42 metres. Two circular flower beds are there on the sides PS and QR with centre at O, the intersections of its diagonals. Find the total area of the two flower beds (shaded parts).



Solution:

PQRS is a square.

So each side is equal and angle between the adjacent sides is a right angle.

Also the diagonals perpendicularly bisect each other.

In Δ PQR using pythagoras theorem,
$PR^2 = PQ^2 + QR^2$
$PR^2 = (42)^2 + (42)^2$
$PR = \sqrt{2}(42)$
$OR = \frac{1}{2}PR = \frac{42}{\sqrt{2}} = OQ$
From the figure we can see that the radius of flower bed ORQ is OR.
Area of sector ORQ = $\frac{1}{4}\pi r^2$
$= \frac{1}{4}\pi\left(\frac{42}{\sqrt{2}}\right)^2$
Area of the Δ ROQ $= \frac{1}{2} \times RO \times OQ$
$= \frac{1}{2} \times \frac{42}{\sqrt{2}} \times \frac{42}{\sqrt{2}}$
$= \left(\frac{42}{2}\right)^2$
Area of the flower bed ORQ
= Area of sector ORQ - Area of the ROQ
$= \frac{1}{4}\pi\left(\frac{42}{\sqrt{2}}\right)^2 - \left(\frac{42}{2}\right)^2$
$= \left(\frac{42}{2}\right)^2\left[\frac{\pi}{2} - 1\right]$
= (441)[0.57]
= 251.37 cm^2
Area of the flower bed ORQ = Area of the flower bed OPS
= 251.37 cm^2
Total area of the two flower beds
= Area of the flower bed ORQ + Area of the flower bed OPS
= 251.37 + 251.37
= 502.74 cm^2

31. From each end of a solid metal cylinder, metal was scooped out in hemispherical from of same diameter. The height of the cylinder is 10 cm and its base is of radius 4.2 cm. The rest of the cylinder is melted and converted into a cylindrical wire of 1.4 cm thickness. Find the length of the wire. [Use $\pi = \frac{22}{7}$]

Solution:
Height of the cylinder (h) = 10 cm
Radius of the base of the cylinder = 4.2 cm
Volume of original cylinder $= \pi r^2 h$
$= \frac{22}{7} \times (4.2)^2 \times 10$
$= 554.4\ cm^3$

Volume of hemisphere $=\frac{2}{3}\pi r^3$

$=\frac{2}{3}\times\frac{22}{7}\times(4.2)^3$

$=155.232\, cm^3$

Volume of the remaining cylinder after scooping out hemisphere from each end

= Volume of original cylinder - 2 $\times$ Volume of hemisphere

$=554.4-2\times155.232$

$=243.936\, cm^3$

The remaining cylinder is melted and converted to
a new cylindrical wire of 1.4 cm thickness.

So they have same volume and radius of new cylindrical wire is 0.7 cm.

Volume of the remaining cylinder = Volume of the new cylindrical wire

$243.936=\pi r^2 h$

$243.936=\frac{22}{7}(0.7)^2 h$

$h=158.4\, cm$

$\therefore$ The length of the new cylindrical wire of 1.4 cm thickness is 158.4 cm.